

# Heavy quarks, gluons and the confinement potential in Coulomb gauge

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**Abstract.** We consider the heavy quark limit of Coulomb gauge QCD, with the truncation of the Yang-Mills sector to include only (dressed) two-point functions. We find that the rainbow-ladder approximation to the gap and Bethe-Salpeter equations is nonperturbatively exact and moreover, we provide a direct connection between the temporal gluon propagator and the quark confinement potential. Further, we show that only bound states of color singlet quark-antiquark (meson) and quark-quark (SU(2) baryon) pairs are physically allowed.

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Coulomb gauge is an ideal choice for investigating the confinement phenomenon. In this gauge, there is an appealing picture of confinement: the Gribov-Zwanziger scenario [1], whereby the temporal component of the gluon propagator provides a long-range confining force whereas the transverse spatial components are infrared suppressed (and therefore do not appear as asymptotic states). Confinement implies the existence of a nonperturbative scale (the string tension), and this naturally leads to the question: is there a simple connection between the (gauge fixed) Green's functions of Yang-Mills theory and this physical scale? To answer this question, we propose to study here the full (nonperturbative) QCD Bethe-Salpeter equation in Coulomb gauge within a heavy quark mass expansion at leading order. In this truncated system, we will use the results inspired by the lattice for the explicit Green's functions of the Yang-Mills sector and in addition we will employ the Slavnov-Taylor identity for the quark-gluon vertex.

We begin by considering the explicit quark contribution to the full QCD generating functional (unless otherwise specified, the Dirac and color indices in the fundamental representation are implicit and we follow the conventions from [2]):

$$Z[\bar{\chi}, \chi] = \int \mathcal{D}\Phi \times \exp \left\{ \iota \int d^4x \bar{q}(x) \left[ \iota \gamma^0 D_0 + \iota \vec{\gamma} \cdot \vec{D} - m \right] q(x) + \iota \int d^4x [\bar{\chi}(x) q(x) + \bar{q}(x) \chi(x)] + \iota \mathcal{S}_{YM} \right\}. \quad (1)$$

$\mathcal{S}_{YM}$  represents the Yang-Mills part of the action and the temporal and spatial components of the covariant derivative (in the fundamental color representation) are

given by

$$D_0 = \partial_0 - \iota g T^a \sigma^a(x), \quad \vec{D} = \vec{\nabla} + \iota g T^a \vec{A}^a(x), \quad (2)$$

where  $\vec{A}$  and  $\sigma$  refer to the spatial and temporal components of the gluon field, respectively.

The full quark field  $q$  ( $\bar{q}$  is the conjugate field, and  $\chi, \bar{\chi}$  are the corresponding sources) is decomposed according to the heavy quark transformation:

$$q(x) = e^{-imx_0} [h(x) + H(x)], \\ h(x) = e^{imx_0} \frac{1 + \gamma^0}{2} q(x), \quad H(x) = e^{imx_0} \frac{1 - \gamma^0}{2} q(x) \quad (3)$$

(similarly for the antiquark field). We now insert this decomposition into the generating functional Eq. (1), integrate out the  $H$ -fields, and make an expansion in the heavy quark mass, similar to Heavy Quark Effective Theory [HQET] [3]. At leading order, we get the following expression:

$$Z[\bar{\chi}, \chi] = \int \mathcal{D}\Phi \exp \left\{ \iota \int d^4x \bar{h}(x) [\iota \partial_0 + g T^a \sigma^a] h(x) + \iota \int d^4x [e^{-imx_0} \bar{\chi}(x) h(x) + e^{imx_0} \bar{h}(x) \chi(x)] + \iota \mathcal{S}_{YM} \right\} + \mathcal{O}(1/m). \quad (4)$$

In the above, we have retained the full quark and antiquark sources (unlike HQET). This means that we can use the full gap and Bethe-Salpeter equations of QCD replacing, however, the kernels, propagators and vertices by their leading order expression in the mass expansion. Also, notice that the spin degrees of freedom have decoupled from the system.

In full Coulomb gauge QCD, the quark gap equation is given by <sup>1</sup>:

$$\begin{aligned}\Gamma_{\bar{q}q}(k) &= \Gamma_{\bar{q}q}^{(0)}(k) + \frac{1}{(2\pi)^4} \int d^4\omega \\ &\times \left\{ \Gamma_{\bar{q}q\sigma}^{(0)a} W_{\bar{q}q}(\omega) \Gamma_{\bar{q}q\sigma}^b(\omega, -k, k-\omega) W_{\sigma\sigma}^{ab}(k-\omega) \right. \\ &\left. + \Gamma_{\bar{q}qAi}^{(0)a} W_{\bar{q}q}(\omega) \Gamma_{\bar{q}qAj}^b(\omega, -k, k-\omega) W_{AAij}^{ab}(k-\omega) \right\}\end{aligned}\quad (5)$$

( $\Gamma$ 's denote the various proper functions,  $W$  denotes propagators, see [4]). The gap equation is supplemented by the Coulomb gauge Slavnov-Taylor identity [2]:

$$\begin{aligned}k_3^0 \Gamma_{\bar{q}q\sigma}^d(k_1, k_2, k_3) &= i \frac{k_{3i}}{k_3^2} \Gamma_{\bar{q}qAi}^a(k_1, k_2, k_3) \Gamma_{\bar{c}c}^{ad}(-k_3) \\ &+ \Gamma_{\bar{q}q}(k_1) \left[ \tilde{\Gamma}_{\bar{q};\bar{c}c q}^d(k_1 + q_0, k_3 - q_0; k_2) + i g T^d \right] \\ &+ \left[ \tilde{\Gamma}_{q;\bar{c}c \bar{q}}^d(k_2 + q_0, k_3 - q_0; k_1) - i g T^d \right] \Gamma_{\bar{q}q}(-k_2)\end{aligned}\quad (6)$$

where  $k_1 + k_2 + k_3 = 0$ ,  $q_0$  is an arbitrary energy injection scale (arising from the noncovariance of Coulomb gauge),  $\Gamma_{\bar{c}c}$  is the ghost proper two-point function,  $\tilde{\Gamma}_{\bar{q};\bar{c}c q}$  and  $\tilde{\Gamma}_{q;\bar{c}c \bar{q}}$  are ghost-quark kernels associated with the Gauss-BRST transform (see also [5]).

Now, as a consequence of the (Coulomb gauge) decomposition, Eq. (3) and under the assumption that the pure Yang-Mills vertices may be neglected (retaining, however, the dressed gluon propagator), the Dyson-Schwinger equation for the nonperturbative spatial quark-gluon vertex furnishes the result that  $\Gamma_{\bar{q}qA} \sim O(1/m)$  (see [2] for a complete discussion and justification of this truncation). Similarly, the ghost-quark kernels can be neglected. Thus, under our truncation scheme, the Slavnov-Taylor identity reduces to

$$\begin{aligned}k_3^0 \Gamma_{\bar{q}q\sigma}^d(k_1, k_2, k_3) &= \\ i g \left[ \Gamma_{\bar{q}q}(k_1) T^d - T^d \Gamma_{\bar{q}q}(-k_2) \right] &+ \mathcal{O}(1/m).\end{aligned}\quad (7)$$

This is then inserted into Eq. (5), together with the tree-level quark proper two-point function

$$\Gamma_{\bar{q}q}^{(0)}(k) = i[k_0 - m] + \mathcal{O}(1/m) \quad (8)$$

and the tree level quark gluon vertex

$$\Gamma_{\bar{q}q\sigma}^{(0)a}(k_1, k_2, k_3) = g T^a + \mathcal{O}(1/m) \quad (9)$$

that follow from the generating functional Eq. (4). The general form of the nonperturbative temporal gluon propagator is given by [6]:

$$W_{\sigma\sigma}^{ab}(\vec{k}) = \delta^{ab} \frac{i}{\vec{k}^2} D_{\sigma\sigma}(\vec{k}^2). \quad (10)$$

Lattice results [7] motivate that the dressing function  $D_{\sigma\sigma}$  is largely independent of energy and likely to behave as  $1/\vec{k}^2$  for vanishing  $\vec{k}^2$  (the explicit form of  $D_{\sigma\sigma}$  will only be needed in the last step of the calculation). Putting all this together, we find the following solution to Eq. (5) for the heavy quark propagator:

$$W_{\bar{q}q}(k_0) = \frac{-i}{[k_0 - m - \mathcal{I}_r + i\epsilon]} + \mathcal{O}(1/m), \quad (11)$$

where the (implicitly regularized, denoted by “ $r$ ”) constant is given by

$$\mathcal{I}_r = \frac{1}{2(2\pi)^3} g^2 C_F \int_r \frac{d\vec{\omega} D_{\sigma\sigma}(\vec{\omega})}{\vec{\omega}^2} + \mathcal{O}(1/m), \quad (12)$$

with the Casimir factor  $C_F = (N_c^2 - 1)/2N_c$ . When solving Eq. (5), the ordering of the integration is set such that the temporal integral is performed first, under the condition that the spatial integral is regularized and finite. Inserting the solution Eq. (11) into the Slavnov-Taylor identity, we find that the temporal quark-gluon vertex remains nonperturbatively bare:

$$\Gamma_{\bar{q}q\sigma}^a(k_1, k_2, k_3) = g T^a + \mathcal{O}(1/m). \quad (13)$$

Note that the propagator Eq. (11) has a single pole in the complex  $k_0$ -plane (due to the mass expansion) and therefore it is necessary to explicitly define the Feynman prescription. From Eq. (11) it then follows that the closed quark loops (virtual quark-antiquark pairs) vanish due to the energy integration, which implies that the theory is quenched in the heavy mass limit:

$$\int \frac{dk_0}{[k_0 - m - \mathcal{I}_r + i\epsilon][k_0 + p_0 - m - \mathcal{I}_r + i\epsilon]} = 0. \quad (14)$$

We also emphasize that the position of the pole has no physical meaning since the quark can never be on-shell. The poles in the quark propagator are situated at infinity (thanks to  $\mathcal{I}_r$  as the regularization is removed) meaning that either one requires infinite energy to create a quark from the vacuum or, if a hadronic system is considered, only the relative energy (derived from the Bethe-Salpeter equation) is important.

The solution for the antiquark propagator reads:

$$W_{q\bar{q}}(k_0) = \frac{-i}{[k_0 + m - \mathcal{I}_r + i\epsilon]} + \mathcal{O}(1/m). \quad (15)$$

Notice the assignment of the Feynman prescription, similar to Eq. (11) – see also the discussion in [2]. This has the important consequence that the Bethe-Salpeter equation for the quark-antiquark states has a physical interpretation of a bound state equation (see below). The corresponding vertex reads:

$$\Gamma_{q\bar{q}\sigma}^a(k_1, k_2, k_3) = -g T^a + \mathcal{O}(1/m). \quad (16)$$

<sup>1</sup> This expression is in the second order formalism and can be directly inferred from the result obtained within the first order formalism [4].

Let us now consider the full homogeneous Bethe-Salpeter equation for quark-antiquark bound states:

$$\Gamma(p; P)_{\alpha\beta} = -\frac{1}{(2\pi)^4} \int dk K_{\alpha\beta;\delta\gamma}(p, k; P) \times [W_{\bar{q}q}(k_+) \Gamma(k; P) W_{\bar{q}q}(k_-)]_{\gamma\delta} \quad (17)$$

where  $k_{\pm}$  (similarly for  $p_{\pm}$ ) are the momenta of the quarks (with the notations from [2]),  $P$  is the 4-momentum of the bound state (assuming that a solution exists),  $K$  represents the Bethe-Salpeter kernel and  $\Gamma$  is the Bethe-Salpeter vertex function for the particular bound state under consideration. Further, we explicitly identify the antiquark contribution, i.e.  $W_{\bar{q}q}(k_-) = -W_{q\bar{q}}^T(-k_-)$  (also in the kernel).

When constructing the Bethe-Salpeter kernel, we use the fact that the temporal integration performed over multiple quark propagators with the same relative sign for the Feynman prescription vanishes (similar to Eq. (14), but in this case the terms originate from internal quark or antiquark propagators) and hence, the kernel reduces to the ladder truncation [2]:

$$K_{\alpha\beta;\delta\gamma}(p, k) = \Gamma_{\bar{q}q\sigma\alpha\gamma}^a W_{\sigma\sigma}^{ab}(\vec{k}) \Gamma_{q\bar{q}\sigma\beta\delta}^{Tb}. \quad (18)$$

We now insert the nonperturbative results for the propagators and vertices, Eqs. (11,13, 15,16) and take the form, Eq. (10), for the temporal gluon propagator. Further, we notice that the Bethe-Salpeter equation is independent of the relative quark energy and hence we can perform the temporal integration over the quark propagators. The energy integration over the quark and antiquark propagators now leads to (unlike Eq. (14)):

$$\int_{-\infty}^{\infty} \frac{dk_0}{[k_+^0 - m - \mathcal{J}_r + i\epsilon][k_-^0 - m + \mathcal{J}_r - i\epsilon]} = \frac{-\pi i}{\mathcal{J}_r}. \quad (19)$$

By using the expression Eq. (12) for  $\mathcal{J}_r$ , and Fourier transforming to coordinate space, we find the following simple energy solution for the pole condition of the Bethe-Salpeter equation Eq. (17):

$$P_0 = \frac{g^2}{(2\pi)^3} \int_r \frac{d\vec{\omega} D_{\sigma\sigma}(\vec{\omega})}{\vec{\omega}^2} \left\{ C_F - e^{i\vec{\omega} \cdot \vec{y}} C_M \right\} + \mathcal{O}(1/m). \quad (20)$$

where  $C_M$  arises from the color structure and is yet to be identified ( $\Gamma$  is not assumed to be a color singlet):

$$[T^a \Gamma(\vec{y}) T^a]_{\alpha\beta} = C_M \Gamma_{\alpha\beta}(\vec{y}). \quad (21)$$

Because the total color charge of the system is conserved and vanishing [8], a single quark (or antiquark) cannot be prepared in isolation. Thus, the bound state energy  $P_0$  can only be either confining for large separations, i.e. increase linearly with the separation between the quark and antiquark, or be infinite when the hypothetical regularization is removed (so that the system cannot

be physically created). If the temporal gluon propagator dressing function is more infrared divergent than  $1/|\vec{\omega}|$ , then

$$C_F = C_M \quad (22)$$

is required such that the spatial integral is convergent. This gives the condition

$$\Gamma_{\alpha\gamma}(\vec{y}) = \delta_{\alpha\gamma} \Gamma(\vec{y}), \quad (23)$$

which means that the quark-antiquark Bethe-Salpeter equation can only have a finite solution for *color singlet* states and otherwise the energy of the system is divergent. Assuming that in the infrared  $D_{\sigma\sigma} = X/\vec{\omega}^2$  (as indicated by the lattice) where  $X$  is some combination of constants, then from Eq. (20) with the condition Eq. (22) we find

$$P_0 \equiv \sigma |\vec{y}| = \frac{g^2 C_F X}{8\pi} |\vec{y}| + \mathcal{O}(1/m). \quad (24)$$

The above result is that there exists a direct connection between the string tension  $\sigma$  and the nonperturbative Yang-Mills sector of QCD at least under the truncation scheme considered here.

A similar calculation performed for the diquark Bethe-Salpeter equation shows that the diquarks are confined for  $N_c = 2$  colors, corresponding to the  $SU(2)$  baryon, and otherwise there are no (finite) physical states.

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